Diffusion in superconducting percolation networks: number of sites visited

LETTER TO THE EDITOR

Diffusion in superconducting percolation networks: number of sites visited

Muhammad Sahimi and Habib Siddiqui

Department of Chemical Engineering, University of Southern California, Los Angeles, CA 90089-1211, USA

Abstract. Two models of diffusion in superconducting percolation networks are studied in a one-dimensional system. We use chains of 5×10^5 sites and study the scaling properties of the mean-squared displacement of 'termites' which execute random walks on the network, mean number of distinct sites visited and mean number of visits to the origin. We find that although these quantities can be described by well defined scaling laws, the values of the critical exponents appear to depend on the time interval in which the walks are studied. We observe at least four different scaling regimes. The implications of these results for higher-dimensional systems are discussed.

The conductivity Σ of an alloy in which a fraction p of metallic bonds are *superconductors* (i.e., bonds with zero resistance) and the rest are *ordinary conductors* (Efros and Shklovskii 1976, Straley 1976) is an important problem which has received considerable attention. As the percolation threshold p_c is approached from below, Σ diverges as $\Sigma \sim (p_c - p)^{-s}$, where s is a universal exponent. The critical behaviour of Σ is the same as that of the dielectric constant of a *metal–insulator* alloy (Efros and Shklovskii 1976, Grannan *et al* 1981, Wilkinson *et al* 1983), and possibly the viscosity of gels (de Gennes 1979), although this latter possibility has been disputed recently (Sahimi and Goddard 1985). The exponent s also appears in the absorption coefficient of random *metal–insulator composites* (Bowman and Stroud 1984). To describe the physics of superconducting networks de Gennes (1980) suggested that we consider a novel form of random walker, called a *termite*, which performs a normal random walk, i.e. one in which the duration of each step is finite, when *off* the superconducting cluster (*on* the ordinary conductors) but which moves instantaneously when *on* the superconducting clusters, because there is zero potential gradient along a superconducting bond. However, because stopping the clock is the only effect of the superconducting clusters, the path of the termite is the same as that of a simple random walk. Therefore, one should expect no singular behaviour for the diffusion coefficient D of the termite except at $p = 1$. Indeed, Bunde *et al* (1985) found for de Gennes's termite that $D = (1 - p)^{-1}$ *for all dimensionalities d.*

Several variations and modifications of de Gennes's termite have been recently discussed. We consider only two of them which are relevant to the present letter and refer the reader to Adler *et al* (1985) and Bunde *et al* (1985) for more details and for a description of other termites. Bunde *et al* (1985) consider a general two-component system where bonds have conductances σ_B with probability p and σ_A with probability $1 - p$. From the Einstein relation, it is easily seen that $\tau_A / \tau_B = \sigma_B / \sigma_A$, where $\tau_A(\tau_B)$ is

the characteristic time to travel a distance l in the region A(B). It is then argued that the transition probability Π_i for choosing the nearest-neighbour site i ($i = 1, \ldots, z$, where z is the coordination number of the network) is given by $\Pi_i = \tau_i^{-1}/\Sigma_i \tau_i^{-1}$, where $\tau_i^{-1} = \sigma_A$ or σ_B depending on whether i is species A or B. If N_A and N_B are the total number of steps in A and B sites, then the total elapsed time is $t = N_A \tau_A + N_B \tau_B$. The limit $\tau_B \to \infty$ and $\tau_A = 1$ describes the termite limit. We refer to this model as the Boston termite 1 (BT1).

In the second model the termite is placed at random onto any site of the network and then chooses to walk to one of the site's z nearest neighbours. If the new site is an ordinary conductor then one unit of time is recorded and the termite moves to this site. If the new site is a superconducting site, then one site of the superconducting cluster to which this site belongs is chosen at random and the termite jumps to this site; no unit of time is added to the total time in this case. So far as modelling of real systems is concerned, this model might be more realistic, because if a particle is on a superconducting cluster, there is no reason for its motion to be restricted to nearest-neighbour jumps; the superconducting material does not offer any resistance to the motion of the particle. This model was recently investigated by Adler *et al* (1985) and it is very similar to what Bunde *et al* (1985) call the Boston termite 2 model. We refer to this as BT2, although it was also called the 'Tel Aviv' termite by Adler *et al* (1985).

In this letter we investigate those aspects of BT1 and BT2 which were not considered by Adler *et al* (1985) and Bunde *et al* (1985). Specifically, we study the scaling behaviour of $S(t)$, the mean number of distinct sites visited at time t, and $M(t)$, the mean number of visits to the origin (of the walk). In general we should expect that

$$S(t) \sim t^\alpha, \tag{1}$$

$$M(t) \sim t^\gamma, \tag{2}$$

and if $\langle R^2(t) \rangle$ is the mean-squared displacement of the termite at time t, one has

$$\langle R^2(t) \rangle \sim t^k. \tag{3}$$

One important question is whether α, γ and k are related. One is also interested in the possible relation between α, γ and k and the static exponents of percolation such as ν, the exponent of percolation correlation length and β, the exponent of percolation probability, and the conductivity exponents s and μ (μ is the exponent of the conductivity of percolation networks of conductors and insulators). A random walk fractal dimensionality d_w' is also defined by $d_w' = 2/k$. If $d_w' = 2$, one has normal diffusion, but for $d_w' \neq 2$ diffusion is anomalous.

We first point out the similarities and differences between the present problem and the 'ant' problem, which is the random walk description of diffusion in percolation networks of conductors and insulators (de Gennes 1976). In the ant problem the walker does not move off the clusters of conducting bonds (sites) since the resistance of the insulating part is infinite. Therefore, depending on whether the random walk takes place on the largest cluster or on *all* clusters with the results averaged over *all* clusters, one would have different expressions for $S(t)$, $M(t)$ and $\langle R^2(t) \rangle$. In the termite problem, the random walk can be started from *any* site of the network and the termite can move to *any* site of the system. Therefore, the termite 'feels' the presence of clusters of different sizes and its motion represents an average over all clusters. Coniglio and Stanley (1984), who studied BT1 in the limit $\tau_B \to \infty$, proposed that

$$d_w' = 2 - s/\nu. \tag{4}$$

However, we believe that (4) is not plausible (except possibly at $d = 1$), because an average over all clusters usually involves the exponent β which is missing from (4). For the ant problem, if an average is taken over all clusters, one obtains (Angles d'Auriac and Rammal 1983)

$$\alpha = (d - 2\beta/\nu)/d_w \tag{5}$$

where $d_w = 2 + (\mu - \beta)/\nu$, with d_w being the fractal dimension of the random walk of the ant. It was recently suggested by Sahimi (1985) that an equation similar to (5) might also hold for the termites with d_w replaced by d'_w. However, the precise form of d'_w and whether it is related to other percolation exponents is currently controversial. One goal of this letter is to check the validity of this hypothesis.

Adler *et al* (1985) studied BT2 in one and two dimensions and argued that instead of calculating $\langle R^2(t) \rangle$, as appropriate for the ant problem, it is much more appropriate to average over t, for fixed R. This was based on the argument that the relevant physical time is not the average time t, for visiting sites at a distance R from the origin, but rather the first exit (passage) time T, at which R is reached for the first time. Such an averaging was first used by Sahimi *et al* (1983) in their study of mixing (diffusion) in flow through random networks near p_c. But as pointed out by Sahimi *et al* (1983) such an averaging is usually appropriate if one has normal diffusion; the validity of such an averaging for anomalous diffusion is not clear to us.

We have studied the BT1 and BT2 models in a one-dimensional system. Although for some problems in critical phenomena one dimension represents a special case, we believe that for the termite problem it is instructive to study it, because in contrast to the ant problem which is trivial at $d = 1$, the present problem is non-trivial and an exact solution is still lacking. Moreover, since all of the percolation exponents at $d = 1$ are known exactly, an hypothesis such as (5) can be tested. As there are similarities between the present problem and some well studied random walk problems, a comparison between these problems can shed light on scaling properties of the termites. We will report the results of our study of the problem in higher-dimensional systems in a future paper.

We used linear chains of 5×10^5 sites with periodic boundary condition. For each independent realisation of the chain we used 200 particles and between 6 to 25 realisations were made. Most of our simulations were done at $p = 0.95$ and 0.97. In any Monte Carlo simulations of diffusion in percolation networks one has to distinguish between the regime in which $R < \xi_p$, and the one in which $R > \xi_p$, where ξ_p is the correlation length. For the ant problem one has $d_w = 2$ ($k = 1$) if $R \gg \xi_p$, whereas $d_w > 2$ if $R < \xi_p$. One might anticipate similar behaviour for the termite problem. However, as we show below, at least in one dimension such behaviour is not observed and $d'_w \neq 2$ even when $R \gg \xi_p$. It is not obvious whether such a phenomenon would also be observed in higher-dimensional systems.

Monte Carlo simulations of the BT2 model in one dimension are very difficult because for p close to unity the random walker can take very long jumps and, therefore, it is difficult to observe the regime $R < \xi_p$ (where ξ_p is of the order of $(1-p)^{-1}$). On the other hand, for lower values of p, ξ_p is very small and for most particles $R > \xi_p$. Thus simulations of this model at $d = 1$ can yield information mainly about the $R > \xi_p$ regime. Very close to $p = 1$ there are very large clusters of superconducting bonds(sites) and the termite can take very long jumps. Therefore, the random walk resembles a ballistic motion. For such a motion it is known that at very long times one has $\langle R^2(t) \rangle \sim t^2$ (i.e., $d'_w = 1$). Indeed, our results presented in figure 1 show that for $t > 10$
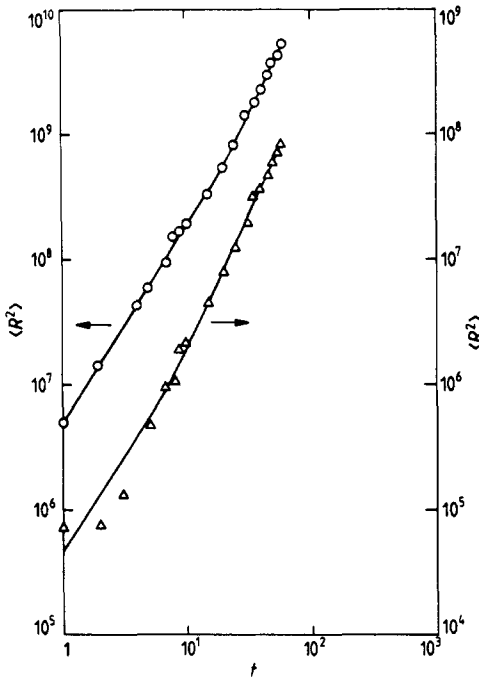
Figure 1. Dependence on time t of the mean-squared displacement $\langle R^2 \rangle$ in the BT2 model. $\bigcirc$, $p = 0.99$; $\triangle$, $p = 0.97$.

both sets of data yield $d'_w \simeq 1$. For $p = 0.99$ we obtain $k \simeq 2.02 \pm 0.07$, while our data for $p = 0.97$ yield $k \simeq 2.047 \pm 0.045$ (the error estimates are statistical only). The result $d'_w = 1$ is consistent with equation (4), although (4) is supposed to hold for $R < \xi_p$, which is not the case here and hence the agreement is perhaps accidental. For $t < 10$ both curves in figure 1 yield $d'_w \simeq \frac{4}{3}$ (although the data for $p = 0.97$ show large fluctuations). Adler *et al* (1985) reported $d'_w \simeq \frac{4}{3}$ *for all times*, whereas we observe at least two distinct scaling regimes.

In figure 2 we present the results for $S(t)$ and $M(t)$ for the BT2 model. Once again there are at least two distinct scaling regimes. For $t > 10$ and at $p = 0.99$ we find $\alpha \simeq 0.97 \pm 0.02$, whereas for shorter times we find $\alpha \simeq 0.75 \pm 0.01$. Similar values are found at $p = 0.97$. It is therefore reasonable to assume that $\alpha = 1$ at long times, whereas $\alpha = \frac{3}{4}$ for shorter times. If this is correct, the results for $S(t)$ would be completely consistent with equation (5), since we found $d'_w = 1$ and $\frac{4}{3}$ for long and short times, respectively. The results for $M(t)$ show similar trends. At long times $M(t)$ approaches a constant value, i.e. $\gamma = 0$, whereas at shorter times we find $\gamma \simeq 0.247 \pm 0.030 \simeq \frac{1}{4}$. These values are consistent with $\gamma = 1 - \alpha$, which holds for the ant problem. The results are also completely consistent with the expressions for $S(t)$ and $M(t)$ which were proposed by Sahimi (1985). We also note that superconducting percolation networks are examples of *antifractals* recently discussed by Pandey (1984). These are disordered systems in which diffusion processes are *faster* than normal (i.e., $d'_w < 2$), in contrast with fractals (e.g., percolation clusters of conducting bonds and lattice animals) in which diffusion processes are *slower* than normal (i.e., $d'_w > 2$).

Monte Carlo simulations of the BT1 model were carried out for the conductivity ratios, $\sigma_B/\sigma_A = 10^2$ and 10^4. The results for $\langle R^2(t) \rangle$ at $p = 0.99$ are presented in figure
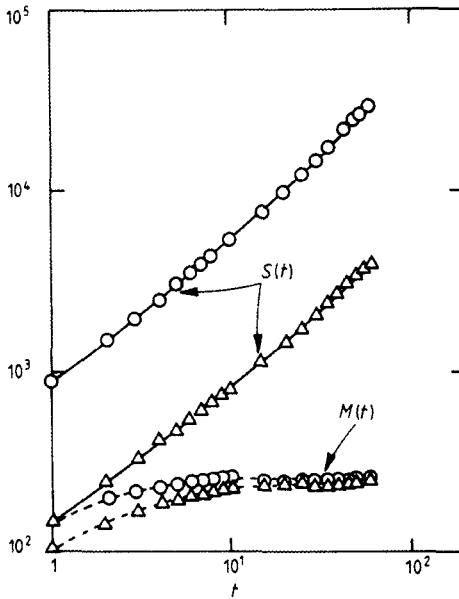
Figure 2. Dependence on time t of the mean number of distinct sites visited $S(t)$ and the mean number of visits to the origin $M(t)$ in the BT2 model. $\bigcirc$, $p = 0.99$; $\triangle$ $p = 0.97$.
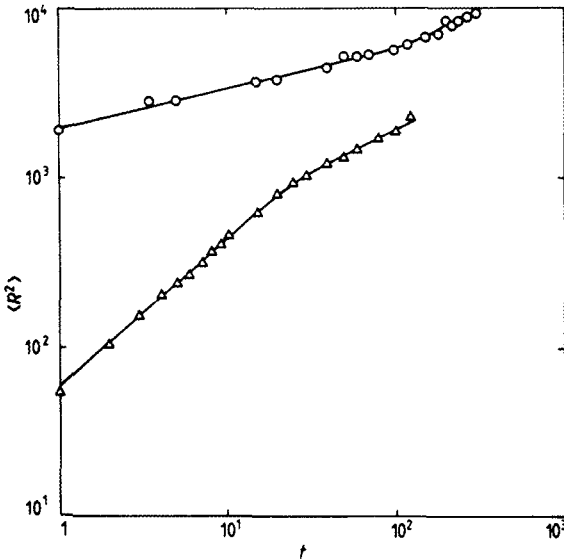

Figure 3. Dependence on time t of the mean-squared displacement $\langle R^2 \rangle$ in the BT1 model at $p = 0.99$. $\bigcirc$, $\sigma_B/\sigma_A = 10^4$; $\triangle$, $\sigma_B/\sigma_A = 10^2$.

3. There are at least two distinct scaling regimes. For $t < 10$ and for $\sigma_B/\sigma_A = 10^4$ we obtain $k \simeq \frac{1}{4}$, whereas for $t > 10$ we find $k \simeq \frac{1}{2}$. Lower or higher values of σ_B/σ_A yield similar results, although the value of k at short times appears to depend on σ_B/σ_A. This is not a totally unexpected result because one expects k to approach 1 as σ_B/σ_A is lowered to unity. The reason for the slow growth of $\langle R^2(t) \rangle$ with t is perhaps as

follows. For p close to unity there are large clusters of good conductors connected to each other by small islands of poor conductors. The termite performs a symmetric random walk within these clusters. If it hits the boundaries of the cluster, since its transition probability for leaving the cluster is much smaller than the probability of being reflected at the boundaries and staying within the cluster, the termite's motion is essentially confined to the original cluster and therefore $\langle R^2(t)\rangle$ grows with t very slowly. At $p = 1$, one has a crossover to the symmetric random walks on a linear chain with $k = 1$.

The results for $S(t)$ and $M(t)$ at $p = 0.99$ and two values of σ_B/σ_A are shown in figure 4. Here $M(t)$ appears to grow with t faster than $S(t)$, in contrast with the BT2
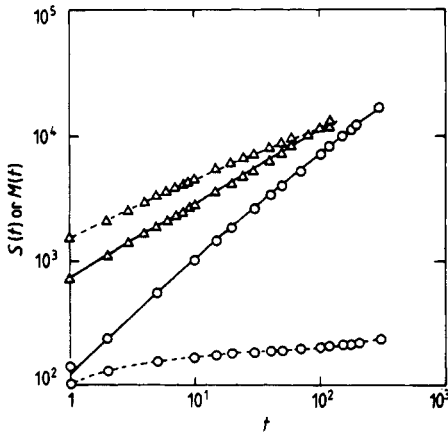

Figure 4. Dependence on time t of the mean number of distinct sites visited $S(t)$ (broken curves) and the mean number of visits to the origin $M(t)$ (full curves) in the BT1 model at $p = 0.99$. $\bigcirc$, $\sigma_B/\sigma_A = 10^4$; $\triangle$, $\sigma_B/\sigma_A = 10^2$.

model and the ant problem. The reason is again in the confinement of the termite to its original cluster with occasional transition to regions of lower conductivity. This means that the termite visits the origin of its motion very frequently, whereas it visits very few new sites after exploring the original cluster. The slopes of the curves in figure 4 appear to depend on σ_B/σ_A, consistent with the behaviour of $\langle R^2(t)\rangle$. The slopes of $S(M)$ increase(decrease) with decreasing σ_B/σ_A since one should have $\alpha = \gamma = \frac{1}{2}$ for $\sigma_B/\sigma_A = 1$. One can also observe at least two scaling regimes for short and long times. The two different regimes disappear as σ_B/σ_A decreases, as expected. The values of α and γ appear again to obey the law $\alpha + \gamma = 1$. For example for $\sigma_B/\sigma_A = 100$ we get $\gamma \simeq 0.58$ and $\alpha \simeq 0.44$ for short times which, within their statistical errors, obey this law. Therefore, this aspect of the termite problem is similar to that of ants.

In summary, we have performed Monte Carlo simulations of two random walk models of diffusion in superconducting percolation networks. The results show that in one dimension the random walk is described by scaling laws whose critical exponents depend on the model and on the time interval in which the walk is studied. This is consistent with the recent work of Bunde and Stauffer (1985). These authors claim that the motion of termites cannot be described in the general case by a two-argument scaling function and one has to use scaling functions with at least three arguments.

Whether the qualitative behaviour of termites found in one dimension persists in higher-dimensional systems remains to be investigated. But we believe that for the BT2 model d'_w is less than two for any dimensionality. We also believe that two- and three-dimensional systems must be studied separately since the duality relation $\mu = s$ which holds in two dimensions may simplify the form of the scaling functions and give the impression that two- and three-dimensional systems may be described by scaling functions with fewer arguments.

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